1 Huygens' Principle

1.1 Light as a wave

In this chapter we no longer consider light and light propagation within the framework of geometrical optics or ray optics, but as a wave propagation phenomen. This translates into a more mathematically description in which the wave nature of light appears. A very basic, but important example is the passage of a light wave through a small aperture in an opaque screen. The ray optics description of this basic experiment leads to the conclusion that the size of the observed lightpattern on an observation screen some distance from the aperture will be simply proportional to the size of the hole, where the opaque screen corresponds to the light shadow.

This is an oversimplification. Such a proportionality law seems to hold quite well for fairly large holes but does not apply at all for smaller holes. Indeed, if we examine carefully the transition from illuminated to nonilluminated areas, the predictions based on the geometrical optics do not hold even for large holes. Furthermore, as the hole area is decreased further, the observed light pattern will increase as the diameter of the hole decreases - an attempt to illustrate this situation is shown in Fig. 1. This figure shows a series of intensity patterns recorded in a plane 30 cm behind a series of apertures illuminated with a collimated beam of quasimonochromatic light. The magnification is 20X and the size of each aperture is indicated, in each case, with its corresponding observed light pattern. Quite clearly, simple geometric predictions are inadequate and fail. A more profound analysis will reveal the wave nature of light which demands a quite different mathematical framework.

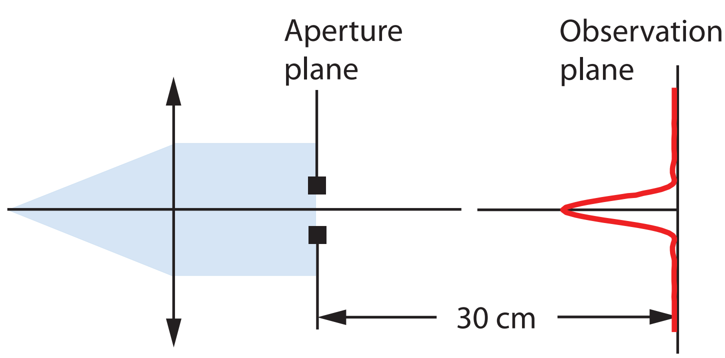

Figure 1: Experimental arrangement for producing diffraction patterns of apertures placed in the aperture plane along with the circular apertures.

From the point of view of basic physics, the wave nature of light is fundamental, stemming from the consideration that light is an electromagnetic phenomen and hence is described by the vector wave equations, which can be derived from Maxwell's equations. This approach to the problem involves a level of mathematical complexity that is out of place in this book. Those interested in this aspect should refer to the books by Saleh & Teich or J. Goodman which presents very readable treatments. For our purposes, we shall assume a more pragmatic approach. We take it as an empirically established fact that a large class of optical phenomena can be accurately described by the hypothesis that light is a scalar, monochromatic wave. For the more fundamental aspects the interested reader may find a profound discussion in J. Goodman's book Introduction to Fourier Optics. At a later point, when we will be in a better position to broaden the discussion and we will look more carefully at this hypothesis.

1.2 Wave propagation

The basic problem of diffraction is the description how a wave propagates from one plane to another. Mathematically this translates into a solution of the wave equation. In its most general form, the scalar wave equation may be written as

$$\nabla^2 V(\vec{\mathbf{x}}, t) = \frac{1}{c^2} \frac{\partial^2 V(\vec{\mathbf{x}}, t)}{\partial t^2}, \tag{1}$$

where $V(\vec{\mathbf{x}}, t)$ is the optical field. For monochromatic waves, $V(\vec{\mathbf{x}}, t)$ separates to a form

$$V(\vec{\mathbf{x}}, t) = \psi(\vec{\mathbf{x}}) e^{i2\pi\nu t}, \tag{2}$$

where $\vec{\mathbf{x}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \vec{\boldsymbol{\xi}} = \xi\mathbf{i} + \eta\mathbf{j} + \chi\mathbf{k}$ [See Eq. (6)], and $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$ are unit vectors along the coordinate axes; ν is the frequency of the wave; and $\psi(\vec{\mathbf{x}})$ describes the spatial variation of the amplitude. By substituting this monochromatic wave into the general wave equation, the time dependence is eliminated and the spatial part of the optical field is seen to satisfy the Helmholtz equation:

$$\nabla^2 \psi + (\frac{2\pi\nu}{c})^2 \psi = 0. \tag{3}$$

At the beginning we will be concerned with the solution of this equation, which can be written in a more convenient form as

$$\nabla^2 \psi + k^2 \psi = 0. \tag{4}$$

Here ψ is the wave field, ∇ is the differential operator, and k is the wave number, $2\pi/\lambda$; where λ is the wavelength. Equation (4) may be rigorously solved using Green's theorem (see, for example, J. Goodman). For the present however, we shall be content with an approximate solution which emphasizes the basic physics of the problem. The relation to the rigorous solution is discussed in Section **??**. Here we shall be concerned with the Huygens' principle development of the solution. Thus, our solution is constructed from the following principle: A **point source** of light will give rise to a spherical wave field emanating equally in all directions. To construct a general solution from this particular one, we note that the Helmholtz equation is linear and hence a *superposition* of solutions is permitted. Next we require only the point of view that an arbitrary wave shape may be considered as a collection of point sources whose strength is given

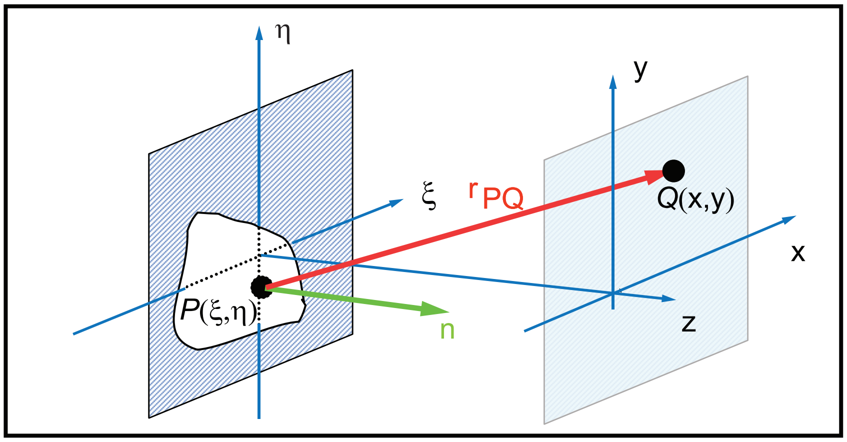

Figure 2: Schematic diagram for wave equation analyses with all pertinent coordinates and distances

by the amplitude of the wave at that point. The field, at any point in space, is simply a sum of spherical waves. This argument, while physically pleasing, ignores the fact that the wave has a preferred direction. We may sharpen the development by the inclusion of an inclination factor to take into account this preferred direction.

This wave field description may be expressed mathematically as follows: a spherical wave is described by the equation

$$\psi_{sp} = \frac{e^{\pm ikr}}{r}, \tag{5}$$

where r is the distance from the point source to the point of observation and the $\pm$ indicates diverging and converging waves, respectively. Thus, if the disturbance across a plane aperture is described by a wave function $\psi'(\vec{\xi})$ (here $\vec{\xi}$ is the position vector in the aperture plane), then the Huygens' principle development for the field at a point $\vec{\mathbf{x}}$ beyond the screen leads to the expression

$$\psi(\vec{x}) = \int_{aperture} \psi'(\vec{\xi})\Lambda(\vec{x},\vec{\xi})\frac{e^{+ikr(\vec{x},\vec{\xi})}}{\mid \vec{r}(\vec{x},\vec{\xi}) \mid} d\vec{\xi}. \tag{6}$$

That is, Eq. (6) simply expresses the fact that a spherical wave of amplitude $\psi'(\vec{\xi})$ emanates from each point $\vec{\xi}$ in the aperture; $\Lambda(\vec{x},\vec{\xi})$ is the inclination factor referred to above. The exact form of this factor will be further developed in a later chapter. For the present, we need only note that $\Lambda(x,\xi)$ is essentially constant if we restrict $\vec{\mathbf{x}}$ and $\vec{\boldsymbol{\xi}}$ to a suitably small region of the neighborhood of the axis, i.e., a line normal to the aperture plane and passing through the center of the aperture. Thus, with this restriction on the observation point, the solution takes the form

$$\psi(\vec{x}) = K \int \psi'(\vec{\xi}) \frac{e^{+ikr(\vec{x},\vec{\xi})}}{\mid \vec{r}(\vec{x},\vec{\xi}) \mid} d\vec{\xi}. \tag{7}$$

The remainder of this chapter is devoted to obtaining solutions of Eq. (7) for various aperture geometries. First we note that the function $\vec{r}(\vec{x},\vec{\xi})$ occurs twice in Eq. (7). The $|\vec{r}|$ in the denominator affects only the amplitude of the wave and may be regarded as a slowly varying function if we restrict $\vec{\mathbf{x}}$ and $\vec{\boldsymbol{\xi}}$ as indicated above. The $\vec{r}$ in the exponent, however, is multiplied by the $k = 2\pi/\lambda$ and affects the phase of the radiation. Hence, while ignoring the effects on amplitude of the variations in r, we must retain the oscillating variation in the exponent caused by the small r variations when integrating over the aperture. Accordingly, Eq. (7) may be further simplified to

$$\psi(\tilde{\mathbf{x}}) = \frac{K}{z} \int \psi'(\vec{\xi}) e^{+ikr(\vec{x},\vec{\xi})} d\vec{\xi}. \tag{8}$$

Since the amplitude varies slowly with the distance r, we may approximate r by z (the screen distance) in the amplitude term. Thus, if we denote the Cartesian coordinates in the aperture by ξ and η, the coordinates in the observation plane by x and y, and the separation of the planes by z, we may use (see Fig. **??**) the Pythagorean theorem and write

$$r(\vec{x},\vec{\xi}) = [(x-\xi)^2 + (y-\eta)^2 + z^2]^{1/2} = R\left[1 + \frac{\xi^2+\eta^2}{R^2} - \frac{2(\xi x + \eta y)}{R^2}\right]^{1/2}, \tag{9}$$

where $R = (x^2+y^2+z^2)^{\frac{1}{2}}$. We now restrict our attention to relatively large distances so that Eq. (9) may be expanded in a binomial series and approximated by its first two terms. Thus,

$$r(\vec{x},\vec{\xi}) \simeq R + \frac{\xi^2+\eta^2}{2R} - \frac{2(\xi x + \eta y)}{2R}. \tag{10}$$

This approximation characterizes most of the optical phenomena in which we are interested. Exceptions [i.e., cases where Eq. (10) is not allowed] are discussed in a later chapter. On the basis of Eq. (10), diffraction problems are customarily divided into groups depending on the relative magnitude of the last two terms. Thus, for those circumstances in which the $(\xi^2+\eta^2)/2R$ may be eliminated in either of the two ways described below.

1.3 Far-Field Approximation

The term $(\xi^2+\eta^2)/2R$ may be eliminated by increasing R to such a point that

$$\frac{k(\xi^2+\eta^2)_{max}}{2R} \ll 1. \tag{11}$$

This condition is called the "far-field" approximation and is of particular importance in many physical optics situations including the design of a pinhole camera since it is exactly this condition that must be met. Using this condition and Eqs. (10) and (8) and noting that $z^2 \gg x^2 + y^2$, we obtain for the field

$$\psi(x, y) = \frac{Ke^{-ikz}}{z} \iint \psi'(\xi, \eta) \exp\left[\frac{-2\pi i}{\lambda z}(\xi x + \eta y)\right] d\xi d\eta, \tag{12}$$

where $z \cong R$. Figures **??**(g) and (h) are examples of this condition.

1.4 Fraunhofer Condition

The Fraunhofer condition

$$k \cdot \frac{\xi^2 + \eta^2}{2R} \ll 1$$

neglects this quadratic term. Experimentally, this condition is achieved by placing a lens in the (ξ, η) plane and observing the diffraction pattern formed at the focus of the lens. To examine this approximation, it is necessary to recall the function of a lens from the point of view of physical optics. Thus, by definition, a lens is a device that converts a plane wavefront into a spherical wavefront of radius f. The concept is illustrated in Fig. 3. Here P is a plane wave incident on the lens; S is a spherical wave emergent from the lens; ρ is the radial height to an arbitrary point on S; and x is the radial distance from the foot of the perpendicular from S to the wavefront. From the Pythagorean theorem we have $(f - x)^2 + \rho^2 = f^2$ or $2xf = \rho^2 - x^2$.

For paraxial optics, x is small and hence x^2 may be ignored by comparison. Thus,

$$x = \frac{\rho^2}{2f} \tag{13}$$

and Eq. (13) is referred to as the "sagittal" approximation. The phase change introduced by a lens is therefore

$$\phi = kx = k\frac{\xi^2 + \eta^2}{2f}. \tag{14}$$

The lens placed in the (ξ, η) plane yields an additional term $exp[-ik(\xi^2 + \eta^2)/2f]$ because it produces a converging spherical wave [see Eq. (5)]. This term cancels the term in $(\xi^2 + \eta^2)$ arising from Eq. (10) when $f = z$. Thus, the field at the point (x, y) in the focal plane is given by

$$\psi(x, y) = \frac{Ke^{-ikf}}{f} \iint \psi'(\xi, \eta) \exp\left[\frac{-2\pi i}{\lambda f}(\xi x + \eta y)\right] d\xi d\eta. \tag{15}$$

Note that Eq.(15) is identical to Eq.(12) if f is substituted for z. These equations state that the field in the far zone or in the focal plane of the lens is the Fourier transform of the field across the diffracting aperture. They constitute the basic

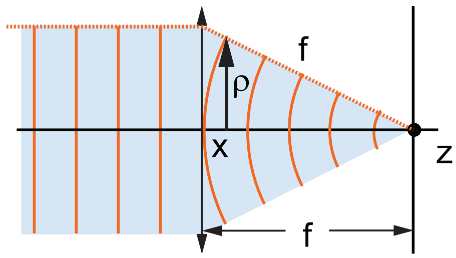

Figure 3: Schematic diagram for lens analysis demonstrating the conversion of a plane wavefront to a spherical wavefront of radius f.

equations of Fraunhofer diffraction theory and are the principal results of this chapter. The use of this integral to determine diffraction patterns is discussed further in later chapters. The experimental realization of these conditions and illustrations will be given.

2 Fourier transform

2.1 Introduction

The principal result of Section1 was the demonstration that the Fraunhofer diffraction pattern associated with the field distribution existing across an aperture is the Fourier transform of that field. To be precise, the Fraunhofer diffraction pattern of an aperture distribution is obtained when the point of observation is very far away, almost infinitely distant from a coherently illuminated aperture. In practice, of course, this condition never describes a physical situation. However, the Fraunhofer theory provides an adequate approximation in many physically significant experiments. These experiments are characterized by one of the following conditions:

1. If the plane containing the **point source** and the **plane of observation** are **parallel conjugate planes** of a well-corrected optical system and both source and point of observation lie near the optical axis, then the Fraunhofer diffraction pattern of the limiting aperture is observed.

2. The distances from the **source to** the diffracting **aperture** z' and from the **diffracting aperture** to the **plane of observation** z are such that

$$|z| \text{ and } |z'| \gg \frac{(\xi^2+\eta^2)_{\max}}{\lambda}, \tag{16}$$

   where ξ and η are coordinates of a general point in the diffracting aperture and λ is the wavelength of the incident wave. Conditions 1 and 2 are those expressed in Section 1 by Eqs. (12) and (15).

The Fourier transform plays an important role in optics, therefore the name Fourier optics was coined. Fourier optics is not only important in the determination of diffraction patterns and the description of interference phenomena, but also in the description of imaging systems and in spectral analysis. Therefore, we will present this topic in its mathematical context as well as an useful demonstration in the context of diffraction theory. Thus, based on Fourier transforms we calculate some basic, but important diffraction patterns. We conclude with a summary of some Fourier transform pairs useful in diffraction calculations.

2.2 Diffraction problems

Under the conditions stated in Sec. 2.1, the amplitude and phase of the field in the focal plane are described by the Fourier transform of the aperture distribution, i.e., apart from constant factors,

$$\psi(x,y) = \exp\left(ik(x^2+y^2)\right) \iint A(\xi,\eta) \exp\left[\frac{-ik}{f}(\xi x + \eta y)\right] d\xi d\eta. \tag{17}$$

Here $A(\xi,\eta)$ corresponds to $\psi'(\xi,\eta)$ used in Sec. 1 and describes the amplitude and phase distribution across the aperture and $\psi(x,y)$ describes the field in the focal plane.[1]

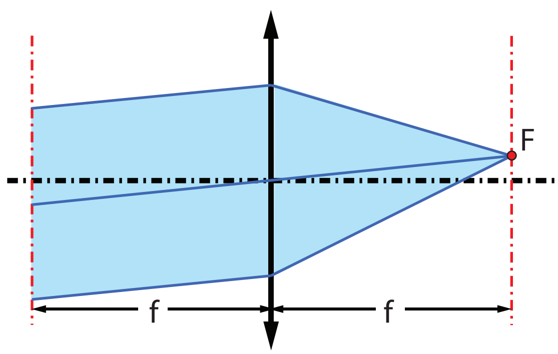

Figure 4: 2-f configuration

Since only the intensity is detected, the quadratic phase term in front of the integral multiplying the Fourier transform in Eq. (17) may usually be neglected. It is important only in the case that the diffraction pattern is allowed to interfere with a coherent background. For the remainder of this chapter, however, the quadratic phase term will be omitted. Thus, the basic formulation of the diffraction problem will take the form

$$\psi(x,y) = \iint A(\xi,\eta) \exp\left[\frac{-ik}{f}(\xi x + \eta y)\right] d\xi d\eta. \tag{18}$$

[1] Note, that we will use a vectorial as well as a scalar description. The vectorial description uses explicit notations as $\vec{\mathbf{x}}$, whereas the scalar description uses (x,y,z) coordinates.

Transform equation (18) is indeed exact in an optical system if the diffracting aperture is in the front focal plane and the recording plane is in the back focal plane of the lens. This corresponds to the so-called 2-f configuration as shown in Fig. 4.

2.2.1 Slit Aperture

As the first example, we consider an infinitely long slit of **width** $2a$ centered along the η axis and uniformly illuminated. We take

$$A(\xi,\eta) = \left\{ \begin{array}{ll} A, & |\xi| < a \\ 0, & |\xi| > a \end{array} \right\} = A \ \mathrm{rect}(\xi|a). \tag{19}$$

This is essentially a one-dimensional problem, where we used the rect- or box-function for the aperture. The diffraction integral can be written as

$$\psi(x) = \int A(\xi) \exp\left(\frac{-ik}{f}(\xi x)\right) d\xi = A \int_{-a}^{a} \exp\left(\frac{-ik\xi x}{f} d\xi\right). \tag{20}$$

The integral in eq. (20) is readily evaluated to give

$$\psi(x) = \frac{-Af}{ikx}\left[\exp\left(\frac{-ikax}{f}\right) - \exp\left(\frac{+ikax}{f}\right)\right] = 2aA\frac{sin\left(\frac{kax}{f}\right)}{\frac{kax}{f}}. \tag{21}$$

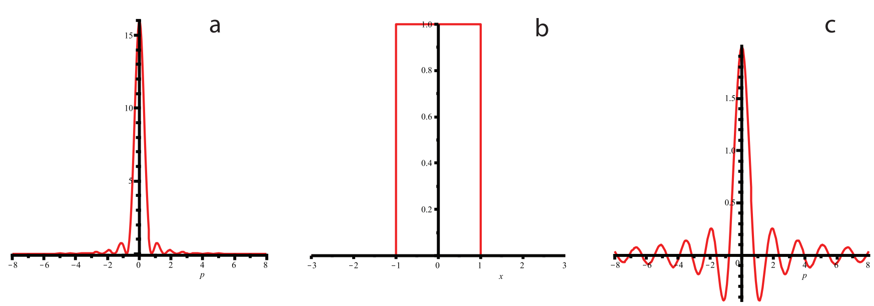

Figure 5: Diffraction by a slit aperture: the main features of the function $\mathrm{sinc}(kax/f)$. a) the intensitiy pattern, b) the slit function c) the diffraction amplitude

The notation $\mathrm{sinc}\theta = sin\theta/\theta$ is frequently used and in terms of this function $\psi(x)$ may be written as

$$\psi(x) = 2aA \ \mathrm{sinc}\frac{kax}{f}. \tag{22}$$

Since we have ignored the quadratic phase term in the calculation of the pattern, Eq. (22) must be interpreted with caution. However, since amplitude and phase are never detected directly in optical experiments, we may neglect this omission except in those cases where $\psi(x)$ is coherently added to another field. Thus, no error is involved in using Eq. (22) to compute the intensity being discussed. The intensity $I(x)$, defined as $\psi(x)\psi^*(x)$, is given by

$$I(x) = 4a^2 A^2 \mathrm{sinc}^2\left(\frac{kax}{f}\right). \tag{23}$$

Figure 5 shows a portion of the function $\mathrm{sinc}(kax/f)$ plotted as a function of x.

2.2.2 Rectangular Aperture

As our second example, we evaluate the diffraction pattern arising from a uniformly illuminated rectangular aperture of width $2a$, length $2b$, and amplitude A centered on the axis of the ξ and η plane. Then

$$\psi(x,y) = A \int_{-a}^{+a}\int_{-b}^{+b} \exp\left[\frac{-ik}{f}(\xi x + \eta y)\right] d\xi d\eta; \tag{24}$$

performing the integration as before we obtain

$$\psi(x,y) = 4\,Aab\ \mathrm{sinc}\left(\frac{kax}{f}\right)\mathrm{sinc}\left(\frac{kby}{f}\right). \tag{25}$$

The intensity is

$$I(x,y) = \psi(x,y)\psi^*(x,y) = 16A^2a^2b^2\mathrm{sinc}^2\left(\frac{kby}{f}\right). \tag{26}$$

2.2.3 Circular Aperture

It is more convenient for this particular example to work in polar coordinates when calculating the diffraction pattern according to eq. (18). We again apply the same conditions for the amplitude and phase across the diffracting aperture. A general point in the diffracting aperture of radius a will have polar coordinates (ρ, ϕ) related to the rectangular coordinates (ξ, η) in the usual way.

$$\xi = \rho\cos\phi,\ \ \eta = \rho\sin\phi. \tag{27}$$

Similarly, a general point in the transform plane has polar coordinates (r, θ). Hence,

$$\begin{aligned}\psi(r,\theta) &= \int_0^a\int_0^{2\pi} A\exp\left(\frac{-ik}{f}r\cos\theta\rho\cos\phi\right)\exp\left(\frac{-ik}{f}r\sin\theta\rho\sin\phi\right)\rho d\rho d\phi \\ &= \int_0^a\int_0^{2\pi} A\exp\left[\frac{-ik}{f}r\cos(\theta-\phi)\right]\rho d\rho d\phi.\end{aligned} \tag{28}$$

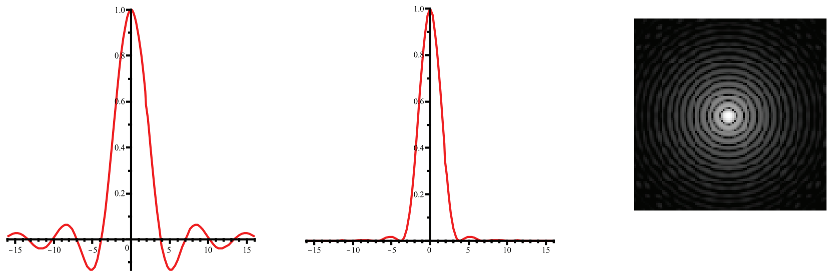

Figure 6: Diffraction by a circular aperture: the main features of the function $2J_1(kar/f)/(kar/f)$.

This integration may be performed using the integral representation of the Bessel function, i.e.,

$$J_n(x) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{-ix\cos\gamma} e^{in\gamma} d\gamma \tag{29}$$

and, in particular,

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ix\cos\gamma} d\gamma. \tag{30}$$

Hence, after a proper substitution, we obtain integration over $d\theta$

$$\psi(r,\theta) = 2\pi A \int_0^a J_0\left(\frac{kr\rho}{f}\right) \rho d\rho. \tag{31}$$

For the Bessel functions we use the recurrence relation

$$\int_0^x x' J_0(x') dx' = x J_1(x), \tag{32}$$

and obtain finally

$$\psi(r,\theta) = A\pi a^2 \frac{2J_1\left(\frac{kar}{f}\right)}{\frac{kar}{f}}, \tag{33}$$

is plotted in Fig.6. The intensity distribution often called the "Airy" pattern, after G. B. Airy who first calculated this diffraction pattern, is given by

$$I(r,\theta) = \psi(r,\theta)\psi^*(r,\theta) = \pi^2 a^4 A^2 \left| \frac{2J_1\left(\frac{kar}{f}\right)}{\frac{kar}{f}} \right|^2. \tag{34}$$

2.3 Fourier Transform and spatial frequencies

Fourier optics provides a general framework for image formation based on harmonic analysis and linear system theory, in brief the notations which have been heuristically introduced in the former sections. So far we have seen, that wave propagation and diffraction is intimately linked to Fourier analysis. However, Fourier analysis is very often perceived as an nonintuitive and rather pure mathematical concept. In the framework of Fourier theory, any signal can be expressed as a sum of a series of so-called harmonic functions. The harmonic function $F(\nu)\exp(j2\pi\nu t)$ with frequency ν and complex amplitude $F(\nu)$ is the basic element of this harmonic analysis. As we have seen in the preceeding section, an arbitrary function $f(t)$ can be decomposed in several basic functions each with its own frequency ν and complex amplitude $F(\nu)$. These harmonic functions play a key role, the fourier transform $F(\nu)$ (also called the fourier transform of $f(t)$) is given as

$$F(\nu) = \int f(t)\exp(-j2\pi\nu t)dt \tag{35}$$

where the integral over t is taken over the range $[-\infty ... +\infty]$. The inverse fourier transform $f(t)$ is given as

$$f(t) = \int F(\nu)\exp(j2\pi\nu t)d\nu \tag{36}$$

where the integral over ν is taken over the range $[-\infty ... +\infty]$, the variable t usually represents time and the conjugated frequency ν has units of cycle/s or Hz. In this time domain description, harmonic analysis allows to expand an arbitrary function of time $f(t)$ as a sum or integral of these harmonic function of time each of them characterised by a specific frequency ν.

Here the meaning "frequency" is mainly used in the context of "temporal frequency", i.e. the frequency denotes a rate of repetition of wave forms in unit time.

Quite similar "spatial frequency" denotes the rate of repetition of a particular pattern in unit distance. The fourier transform becomes now a function over two spatial coordinates (x, y). For imaging the fourier analysis is generalized to functions of two variables i.e. a function $f(x, y)$ may be decomposed as a superposition integral of spatial harmonic functions in x and y. The inverse fourier transform is given as

$$f(x, y) = \iint F(\nu_x, \nu_y)\exp(j2\pi(\nu_x x + \nu_y y))\, d\nu_x d\nu_y \tag{37}$$

where the double integral over (x, y) extends over the ranges $[-\infty ... +\infty]$. The fourier transform $F(\nu_x, \nu_y)$ is given as

$$F(\nu_x, \nu_y) = \iint f(x, y)\exp(-j2\pi(\nu_x x + \nu_y y))\, dxdy \tag{38}$$

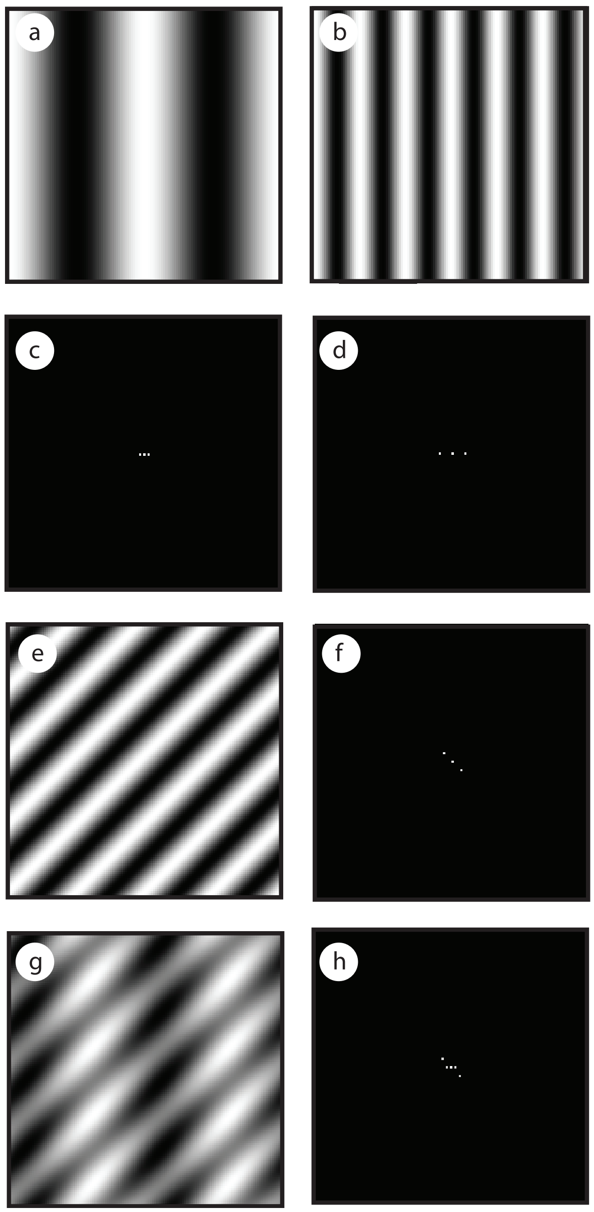

Figure 7: periodic patterns a) low frequency pattern b) high frequency pattern c) the fourier transform of (a) d) the fourier transform of (b) e) a tilted periodic pattern f) the fourier transform of (e) g) adding pattern (a) and (e) h) the fourier transform of (g)

The spatial frequencies ν_x, ν_y are indispensable in a quantitative description of image formation and a main element of Fourier optics. For example a periodic pattern as shown in fig. 7 can be captured in a single Fouier term that encodes 1: the spatial frequency, 2: the magnitude (positive or negative), and 3: the phase.

Quite obviously the spatial frequency corresponds to the periodicity, the magnitude to the brightness and the phase to the spatial placement of individual stripes. The last expample in fig. 7 shows a linear superposition of two spatial frequencies. This adding is nothing else than an image synthesis i.e. or mathematically the superposition of elementary harmonic functions.

2.3.1 Example - the infinite slit

In the former section (2.2.1) we applied our fourier transform analysis to a slit opening. Reregarding this calculation within the harmonic analysis means that we decompose the slit opening in its harmonic frequency content. It is straightforward to name this frequency content the object spectrum of our slit object.

Fig. 8 shows the fourier analysis of this infinit slit. The upper part show the fundamental pattern of frequency 1, together with the higher harmonics. These are the "odd harmonics" which exhibit a bright vertical band in the center. The corresponding fourier transforms are shown below. In the following the adding of these harmonics is given. This superposition of harmonics leads to a bright and brighter slit in the image center. The lower part of the figure shows what would happen if this process were continued, it would produce a thin vertical stripe of high brightness with sharp boundaries. The fourier transform of this opening exhibits an "infinite"series of harmonics, the spectrum of this object is indicated in the figure beside. Quite obviously this figure corresponds to the calculated amplitude spectra (eq. 22) or precisly its corresponding intensity pattern as expressed by eq. 23.

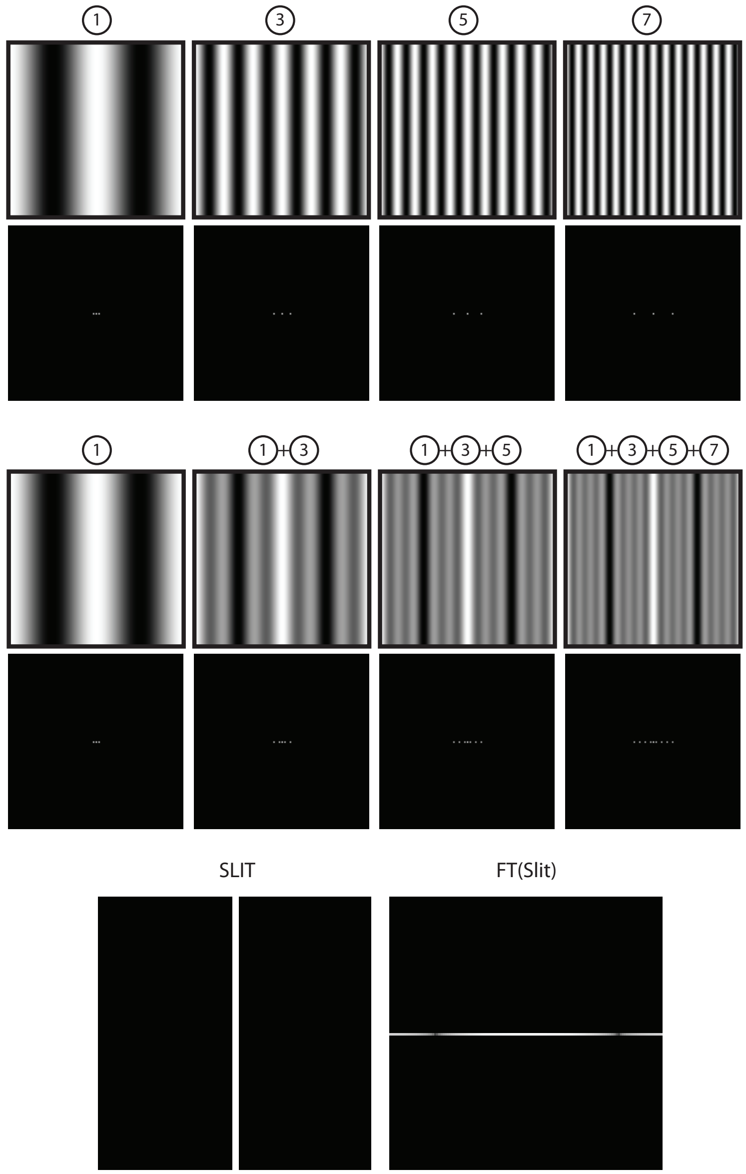

Figure 8: the infinite slit and its fourier decomposition

2.4 Plane wave and harmonic function

Our consideration of wave propagation and diffraction has been based on the Helmholtz equation (compare eq. 4)

$$\nabla^2\psi + k^2\psi = 0.$$

The simpelst solution to the Helmholtz equation in homogenous medium are the plane wave as well as the spherical wave (see eq. 5). The plane wave has the complex amplitude

3 Array Theorem

3.1 Introduction

In this chapter, we continue the study of Fraunhofer or far-field diffraction. As we have seen before, Fraunhofer diffraction leads to a diffracted amplitude distribution proportional to the Fourier transform of the amplitude distribution across the diffracting aperture (see Fig. 9). In the former chapters, some examples of diffraction calculations were given for the most important and standard aperture shapes. In this chapter, we examine in particular the effect of combining apertures of similar shape. In this case, the diffraction integral assumes an interesting and characteristic form and gives rise to a subclass of diffraction effects that is important enough to receive a special nomenclature and study, namely, interference by division of a wavefront.

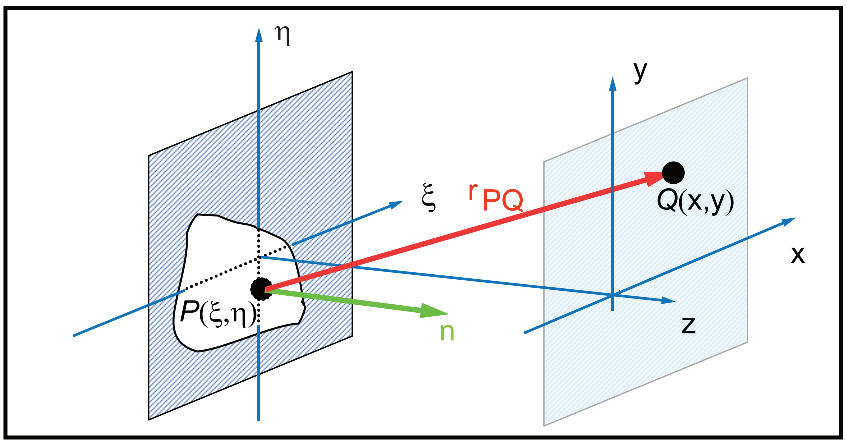

Figure 9: Kirchhoff Integral and diffraction

3.2 The array theorem

A large number of interference problems involve the combination of similar diffraction patterns. They arise in the study of the combined diffraction patterns of an array of similar diffracting apertures. This entire class of interference effects can be described by a single relation, the so-called array theorem. This unifying theorem is the underlying mathematical concept for studing interference by wavefront division.

Let $\psi(\vec{\xi})$ represent the amplitude and phase distribution across **one** aperture centered in the diffraction plane, and let the **total** diffracting aperture consist of a collection of these elemental apertures at different locations $\vec{\xi}_n$. This concept is illustrated in Fig. 10. We require first a method of representing such an array. The appropriate representation is obtained readily by means of the delta function. Thus, if an elemental aperture is positioned such that the location vector is at the point $\vec{\xi}_n$, the appropriate distribution function is $\psi(\vec{\xi} - \vec{\xi}_n)$.

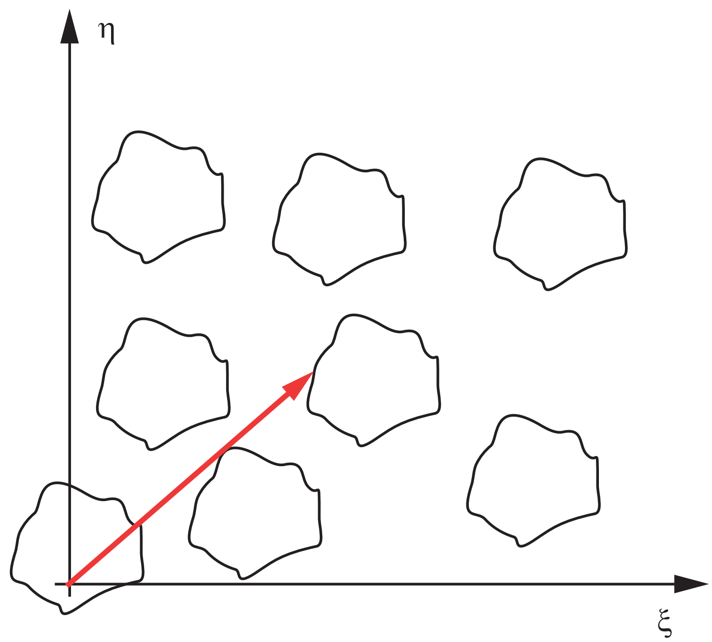

Figure 10: Array theorem - handling a distribution of identical openings

The combining property of the delta function allows us to represent this distribution as follows:

$$\psi(\vec{\xi} - \vec{\xi}_n) = \int \psi(\vec{\xi} - \vec{\alpha})\delta(\vec{\alpha} - \vec{\xi}_n)d\vec{\alpha}. \tag{39}$$

The integral in Eq. (39) is a convolution integral and plays an important role in Fourier optics. Thus, if we wish to represent a large number N of such apertures with different locations, we could write the total aperture distribution $\Psi(x)$ as a sum, i.e.,

$$\Psi(\vec{\xi}) = \sum_{n=1}^{N} \psi(\vec{\xi} - \vec{\xi}_n). \tag{40}$$

Or by using the combining property of the delta function i.e. combining the features of Eqs. (40) and (39),

$$\Psi(\vec{\xi}) = \sum_{n=1}^{N} \psi(\vec{\xi} - \vec{\alpha})\delta(\vec{\alpha} - \vec{\xi}_n)d\alpha. \tag{41}$$

Eq. (41) may be put in a more compact form by introducing for the locations

$$A(\vec{\alpha}) = \sum_{n=1}^{N} \delta(\vec{\alpha} - \vec{\xi}_n), \tag{42}$$

whence Eq.(41) becomes

$$\Psi(\vec{\xi}) = \int \psi(\vec{\xi} - \vec{\alpha}) A(\vec{\alpha}) d\vec{\alpha}. \tag{43}$$

which is physically interesting in the sense that $A(\vec{\alpha})$ characterizes the array itself. That is, $A(\vec{\alpha})$ describes the location of the apertures and $\psi(\vec{\xi})$ describes the distribution across a single aperture. We can now calculate the Fraunhofer diffraction pattern associated with this array. From the former chapter on Huygens principle , we obtained that the Fraunhofer pattern is the Fourier transform of the aperture distribution. Thus, the Fraunhofer pattern $\tilde{\Psi}(\vec{x})$ of the distribution $\Psi(\vec{x})$ is given by $\Psi(\vec{\xi})$

$$\tilde{\Psi}(\vec{x}) = \int \Psi(\vec{\xi}) \exp(\frac{-2\pi i \vec{\xi} \cdot \vec{x}}{\lambda f}) d\xi \tag{44}$$

substituting from Eq. (43) gives

$$\tilde{\Psi}(\vec{x}) = \left[\int \int \psi(\vec{\xi} - \vec{\alpha}) A(\vec{\alpha}) d\vec{\alpha} \right] \exp(\frac{-2\pi i \vec{\xi} \cdot \vec{x}}{\lambda f}) d\vec{\xi}. \tag{45}$$

A very important theorem of Fourier transforms (a proof is given in the Appendix, Sec. 4.2) states that the Fourier transform of a convolution is the product of the individual Fourier transforms. Thus, Eq. (45) may be written as a simple product

$$\tilde{\Psi}(\vec{x}) = \tilde{\psi}(\vec{x}) \tilde{A}(\vec{x}), \tag{46}$$

where $\tilde{\psi}(\vec{x})$ and $\tilde{A}(\vec{x})$ are the Fourier transforms of $\psi(\vec{\xi})$ and $A(\vec{\alpha})$. Equation (46) is the **array theorem** and states that the diffraction pattern of an array of identical apertures is given by the product of the elemental pattern $\tilde{\psi}(\vec{x})$ and the pattern that would be obtained by a similar array of point sources, $\tilde{A}(\vec{x})$. Thus, the separation that first arose in Eq. (43) is retained. To analyze the complicated patterns that arise in interference problems of this sort, one may analyze separately the effects of the array and the effects of the individual apertures.

3.3 Applications of array theorem

3.3.1 Two-Beam Interference

In this section, we use Eq. (46) to describe the simplest of interference experiments, Young's double-slit experiment in **one** dimension. To facilitate interpretation of the results, the transform is written in the sharpened form as given according to the Huygens principle. Thus, the individual aperture will be described by

$$\psi(\xi) = \left\{ \begin{array}{ll} C, & |\xi| \leq a \\ 0, & |\xi| > a \end{array} \right\} = \mathrm{rect}(\xi|a). \tag{47}$$

Here C is a constant representing the amplitude transmission of the apertures.

From the former chapter on Fourier transforms we have the result that the elemental distribution in the Fraunhofer plane is

$$\tilde{\psi}(x) = 2aC \ \mathrm{sinc}\frac{2\pi a x}{\lambda f}. \tag{48}$$

The array in this case is simply two delta functions; thus,

$$A(\xi) = \delta(\xi - b) + \delta(\xi + b). \tag{49}$$

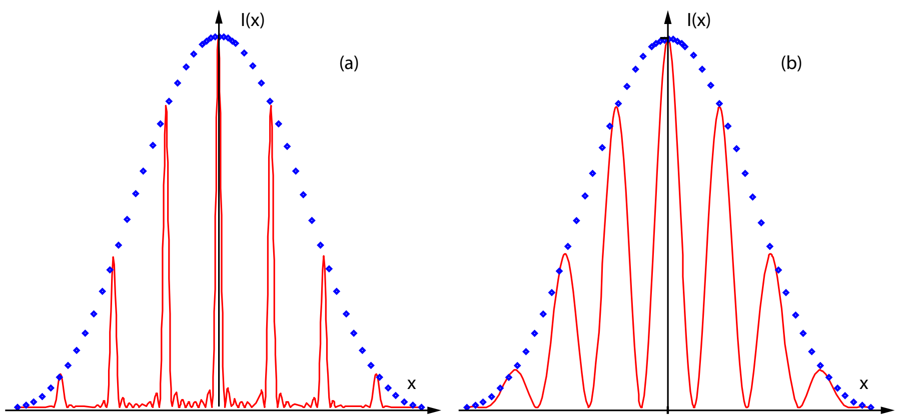

Figure 11: (a) The normalized intensity distribution in the Fraunhofer pattern of two apertures having the ratio $b/a \simeq 4.5$ and (b) the normalized intensity distribution in the Fraunhofer pattern of a larger number of apertures having the same ratio as (a).

The array pattern is, therefore,

$$\tilde{A}(x) = \int [\delta(\xi - b) + \delta(\xi + b)] \exp\left(\frac{-2\pi i \xi x}{\lambda f}\right) d\xi; \tag{50}$$

Eq. (50) is readily evaluated by using the combing property of the delta function. Thus,

$$\tilde{A}(x) = \exp\left(\frac{2\pi i b x}{\lambda f}\right) + \exp\left(\frac{-2\pi i b x}{\lambda f}\right) = 2\cos\left(\frac{2\pi b x}{\lambda f}\right). \tag{51}$$

Finally, the diffraction pattern of the array of two slits is

$$\tilde{\Psi}(x) = 4aC \ \mathrm{sinc}\left(\frac{2\pi i b x}{\lambda f}\right) \cos\left(\frac{2\pi b x}{\lambda f}\right) \tag{52}$$

The intensity is

$$I(x) = 16a^2C^2 \operatorname{sinc}^2\left(\frac{2\pi ibx}{\lambda f}\right)\cos^2\left(\frac{2\pi bx}{\lambda f}\right) \tag{53}$$

From Eq. (53) it is clear that the resulting pattern has the appearance of cosine^2 fringes of period $\lambda f/b$ with an envelope $\operatorname{sinc}^2(2\pi ax/\lambda f)$. A typical distribution is shown in Fig. 11(a).

In a precisely similar manner, we can use our previous results to build up the expressions for the interference observed by using two square apertures and two circular apertures. It is suggested that the reader solve these two problems as an exercise in the use of the array theorem.

3.3.2 One-Dimensional Array

For N identical apertures equally separated by a distance $2b$, the array theorem takes the general form derived below, of which the two-beam example is a special case. The array of delta functions will be represented by a sum of the form

$$A(\xi) = \sum_{n=0}^{N-1} \delta(\xi - 2nb). \tag{54}$$

Thus, the Fourier transform of the array is given by

$$A(\xi) = \sum_{n=0}^{N-1} \exp\left(\frac{-2\pi i2bxn}{\lambda f}\right), \tag{55}$$

which may be written as (geometrical series)

$$= \frac{1-\exp\left(\frac{-2\pi iN2bx}{\lambda f}\right)}{1-\exp\left(\frac{-2\pi i2bx}{\lambda f}\right)}. \tag{56}$$

Therefore,

$$\tilde{\psi}(x) = 2aC\left[\frac{1-\exp\left(\frac{-2\pi iN2bx}{\lambda f}\right)}{1-\exp\left(\frac{-2\pi i2bx}{\lambda f}\right)}\right]\operatorname{sinc}\left(\frac{2\pi \mathrm{ax}}{\lambda \mathrm{f}}\right)$$

and

$$I(x) = 4a^2C^2\left[\frac{1-\cos\frac{N2\pi2bx}{\lambda f}}{1-\cos\frac{2\pi2bx}{\lambda f}}\right]\operatorname{sinc}^2\left(\frac{2\pi \mathrm{ax}}{\lambda \mathrm{f}}\right) \tag{57}$$

Hence, rewriting the term in square brackets,

$$I(x) = 4a^2C^2\left[\frac{\sin^2\left(\frac{2\pi Nbx}{\lambda f}\right)}{\sin^2\left(\frac{2\pi bx}{\lambda f}\right)}\right]\operatorname{sinc}^2\left(\frac{2\pi \mathrm{ax}}{\lambda \mathrm{f}}\right) \tag{58}$$

Note that Eq. (58) reduces to Eq. (53) when $N = 2$. Figure 11(b) shows the distribution for N large.

3.4 Some examples

In this section, a number of illustrations are given covering the mathematical discussions on Fourier transforms and the preceding discussion of the array theorem.

Figure 12 shows a series of photographs of the Fraunhofer diffraction patterns of single apertures. The apertures are shown, with the correct orientation, below the photographs.

Figure 13 illustrates diffraction by an array of two to six slits. The whole of these patterns is contained in the first maximum of the sinc^2 envelope function. Note that, as the number of slits increases, the main maxima stay in the same position but get sharper and there are $N-2$ subsidiary maxima.

Finally, Fig. 14 illustrates some further examples of the array theorem.

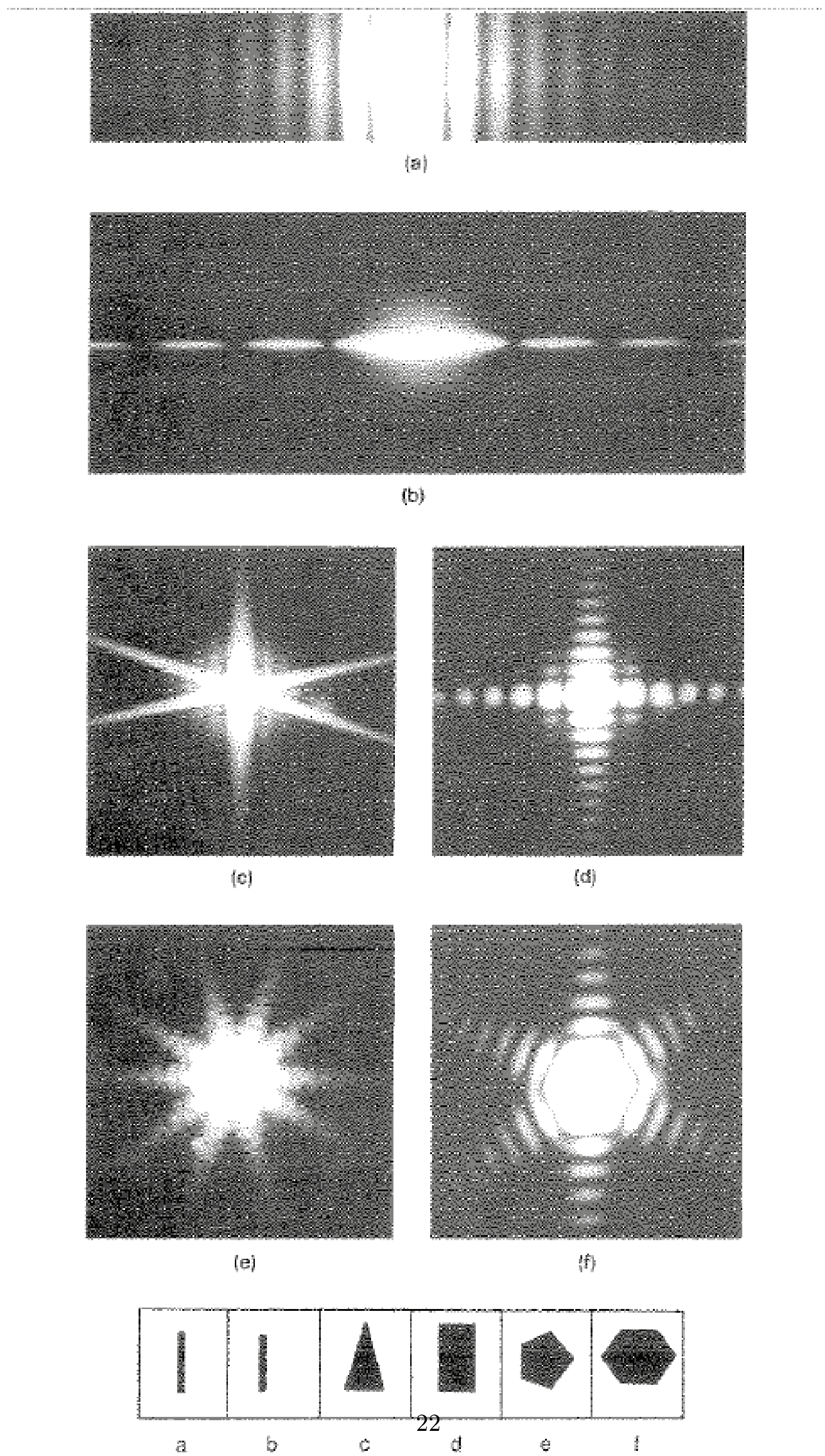

Figure 12: Photographs of the Fraunhofer diffraction patterns (Fourier transforms) of various apertures: (a) slit aperture with a slit source, (b) slit aperture with point source, (c) triangular aperture, (d) rectangular aperture, (e) pentagonal aperture, and (f) hexagonal aperture.

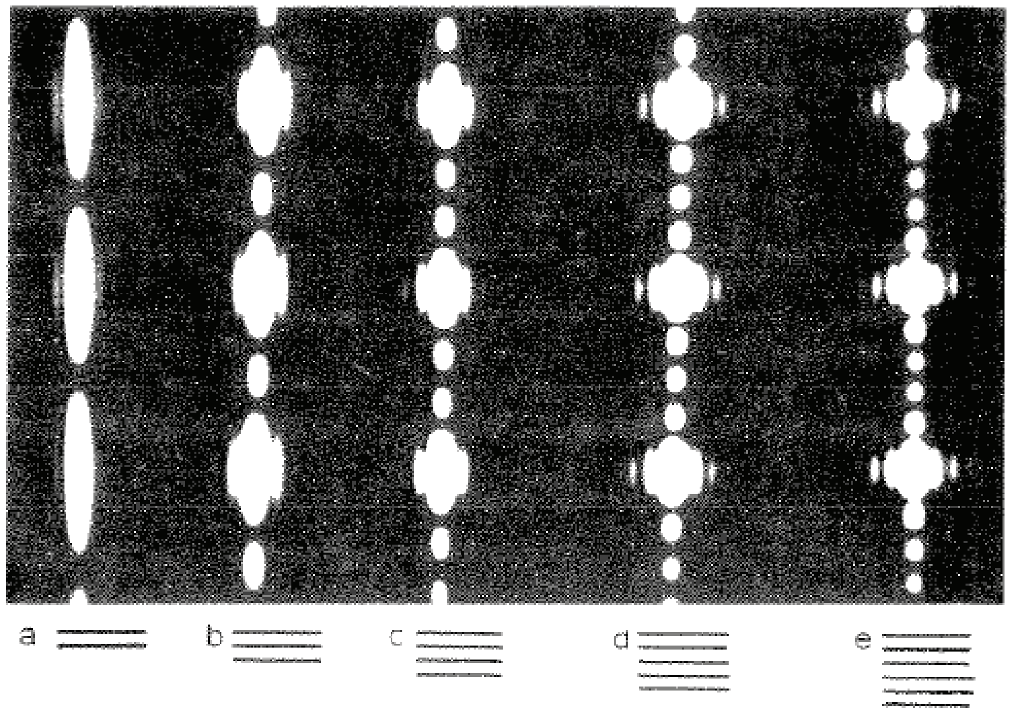

Figure 13: Diffraction patterns of various numbers of slits from two to six, with point source

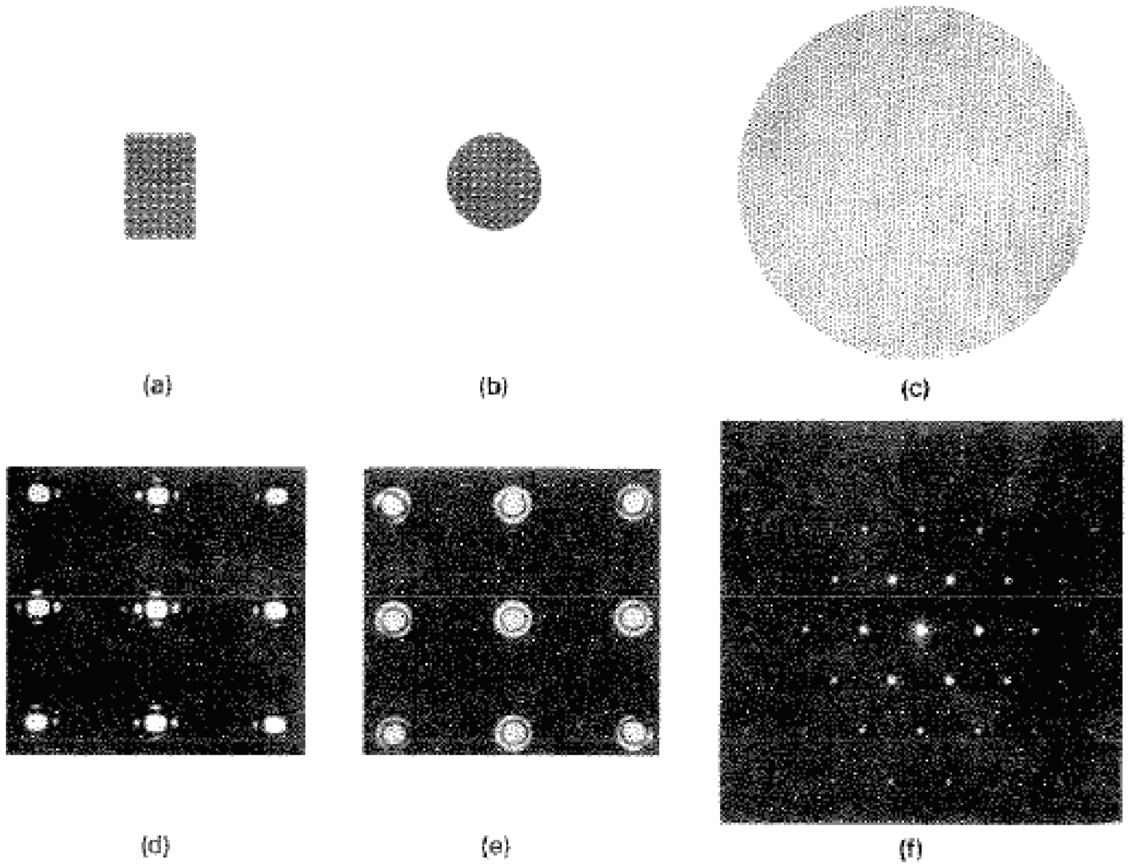

Figure 14: Illustration of the array theorem showing (a) an array of circular apertures contained within a rectangular aperture, (b) an array of circular apertures contained within a circular aperture, (c) a hexagonal array of circular objects and (d), (e), and (f), respectively, the corresponding Fraunhofer diffraction patterns.

3.5 Conclusion

As a shorthand notation, we introduce the tilde, to denote the spatial Fourier transform. This notation is used throughout the remainder of this book; i.e.,

$$\tilde{\psi}(x,y) = \int \psi(\xi,\eta) e^{-ik(\xi x+\eta y)} d\xi d\eta. \tag{59}$$

In conclusion we should like to collect a number of Fourier transform pairs to form a table as shown in Fig. 3.5. In each case, the function $\psi(\xi,\eta)$ transforms to the function $\tilde{\psi}(x,y)$ and vice versa. The table is self explanatory and

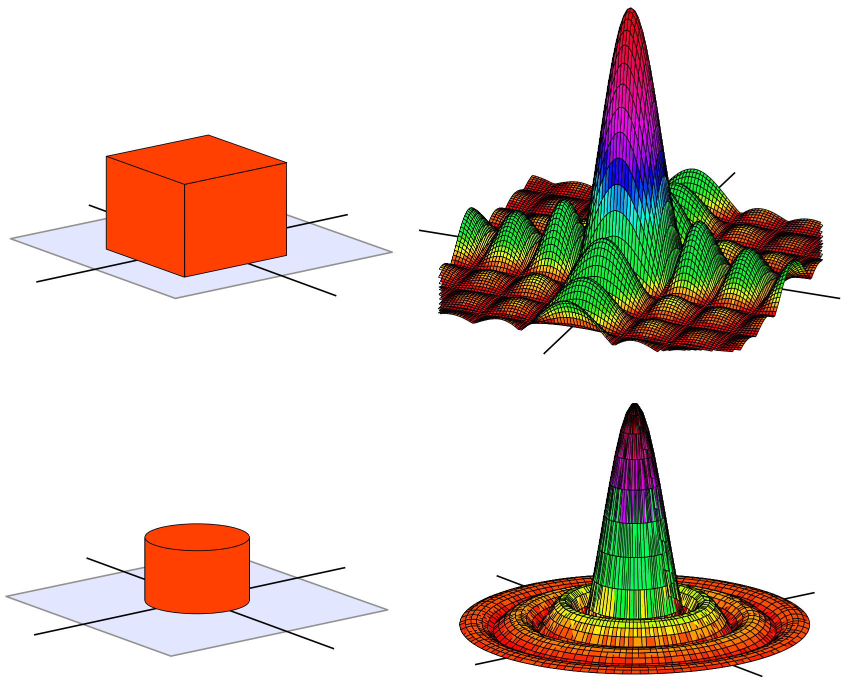

Figure 15: 3D_fourier transform pairs of particular interest in diffraction

needs no further comment, except to state that the curves are diagrammatic representations only.

4 Appendix

4.1 Delta Function

In many diffraction and interference problems, it proves convenient to make use of a special function, the Dirac delta function. This function is defined by the following property: let $f(\xi)$ be any function (satisfying some very weak convergence conditions) and let $\delta(\xi - \xi')$ be a delta function centered at the point ξ'; then

$$\int_a^b f(\xi)\delta(\xi - \xi')d\xi = \left\{ \begin{array}{ll} f(\xi'), & (a < \xi' < b) \\ 0, & \text{otherwise} \end{array} \right\}. \tag{60}$$

We note, therefore, that

$$\int_{-\infty}^{\infty} \delta(\xi - \xi')d\xi = 1. \tag{61}$$

The Fourier transform of the delta function is given by

$$\psi(x) = \int \delta(\xi - \xi') \exp\left(\frac{-ikx\xi}{f}\right) d\xi, \tag{62}$$

which by the definition of the delta function becomes

$$\psi(x) = \exp\left(\frac{-ikx\xi'}{f}\right). \tag{63}$$

The amplitude is constant and the phase function $(kx\xi'/f)$ depends on the origin. When $\xi' = 0$ then the delta function is at the origin and the transform is a constant. The converse is also true, of course, that a constant extending from $-\infty$ to $+\infty$ transforms to a delta function.

4.2 convolution theorem

The derivation of the convolution theorem is greatly facilitated by making use of an integral representation of the delta function. A common and useful **one**-dimensional representation may be obtained by taking the Fourier transform of both sides of Eq. (60) in the preceding chapter. It is

$$\delta(x - x') = \int \exp^{-2\pi i\mu(x-x')} d\mu. \tag{64}$$

This representation is given without proof, more information can be found in the literature (see further reading).

Let $f(x)$ and $g(x)$ be two functions that possess Fourier transforms, i.e.,

$$\tilde{f}(\mu) = \int f(x) e^{2\pi i\mu x} dx, \tag{65}$$

$$\tilde{g}(\mu) = \int fg(x) e^{2\pi i\mu x} dx. \tag{66}$$

By the inversion theorem, we write

$$f(x) = \int \tilde{f}(\mu) e^{-2\pi i\mu x} d\mu, \tag{67}$$

$$g(x) = \int \tilde{g}(\mu) e^{-2\pi i\mu x} d\mu. \tag{68}$$

We wish to find an expression for the Fourier transform of $h(x) = f(x)g(x)$. Thus,

$$\begin{aligned} \tilde{h}(\mu) &= \int h(x) e^{2\pi i\mu x} dx \\ &= \int f(x) g(x) e^{2\pi i\mu x} dx \end{aligned} \tag{69}$$

Substituting from Eqs. (67) and (68) into Eq. (69) and introducing the dummy variables τ and η to avoid the confusion of too many μ's, we obtain

$$\tilde{h}(\mu) = \int \left[\iint f(\tau)g(\eta)e^{-2\pi\tau x}e^{-2\pi\eta x}d\tau d\eta \right] e^{2\pi i\mu x}dx \tag{70}$$

Rearranging the integrals in Eq. (70), we write

$$\tilde{h}(\mu) = \iiint f(\tau)g(\eta)e^{2\pi x[\mu-(\tau+\eta)]}dxd\tau d\eta. \tag{71}$$

The innermost integral (running over x) in Eq. (71) is just a Dirac delta function according to Eq. (64); thus, Eq. (71) reduces to

$$\tilde{h}(\mu) = \iiint f(\tau)g(\eta)\delta(\tau-(\mu-\eta))dxd\tau d\eta. \tag{72}$$

Using the combing property of the delta function,

$$\tilde{h}(\mu) = \int f(\mu-\eta)g(\eta)d\eta. \tag{73}$$

Equation (73) is the convolution theorem and expresses the Fourier transform of $h(x)$ in terms of the convolution of the transforms of its products. The inversion theorem applied to Eq. (73) gives us the convolution theorem in the form required for the array theorem. Thus,

$$\begin{aligned} h(x) = f(x)g(x) &= \int \tilde{h}(\mu)e^{-2\pi\mu x}d\mu \\ &= \int \left[\int f(\mu-\eta)g(\eta)d\eta \right] e^{-2\pi\mu x}d\eta. \end{aligned} \tag{74}$$

There are, of course, other ways of deriving this identity but we have purposely chosen the method using the delta function as a further illustration of its use.

5 Impulse Response

5.1 Introduction

Based on these wave-optical considerations we will now derive physical optical aspects of image formation. This approach will show us some tight links between important parameters known from geometrical optics as object and image distances, magnification, etc. More important, however, is the fact that image quality limitations are immediately evident from such a development. The word "quality" is used here to describe the subjective impression.

Let us start the problem of image formation from the physical optical point of view by considering objects that are incoherently illuminated.

5.2 Impulse response

To give a completely general and consistent development of image formation with incoherent light requires several important results from the theory of partial coherence. As we have not yet discussed this theory, we resort to a two-step development in this chapter. This development, while correct and physically meaningful, is not detailed enough to include the subtleties that can be treated later. We are concerned with incoherent quasimonochromatic light. The precise definition of the term *quasimonochromatic* requires some concepts from coherence theory and, therefore, is rigorously defined in a later Section **??**. Here it is sufficient to imagine a quasimonochromatic light field as a light field, where its frequency bandwith is a small fraction of its central frequency and which is used in experiments involving sufficiently small path differences. Without the rigor of coherence theory, we have to characterize such a system by single concepts so far unexplained until a proper treatment of coherence theory. This concept for incoherent imaging may be simply stated: An optical system employing incoherent radiation may be regarded as linear and stationary in intensity. This statement implies two properties: linearity and stationarity, which are explained in the following sections. For mathematical simplicity, the development in this chapter is in **one** dimension. In later chapters, the analysis is extended to two dimensions where necessary.

5.2.1 Linearity

A system is linear if the addition of inputs produces an addition of corresponding outputs. Throughout this discussion, the arrow, $\rightarrow$, should be read as "produces". If $f_1(x')$ is an input that produces an output $g_1(x)$ denoted by

$$f_1(x') \rightarrow g_1(x), \tag{75}$$

and if

$$f_2(x') \rightarrow g_2(x), \tag{76}$$

then

$$af_1(x') + bf_2(x') \rightarrow ag_1(x) + bg_2(x), \tag{77}$$

where a and b are constants.

5.2.2 Stationarity

The property of stationarity implies that if the location of the input is changed, i.e., $f_1(x')$ is replaced by $f_1(x'-x'_0)$, the only effect on the output is to change its location, i.e.,

$$f_1(x'-x'_0) \rightarrow g_1(x-x_0). \tag{78}$$

These properties of linearity and stationarity are fundamental to the development of image theory. Under suitable conditions, optical systems possess these properties.

To develop an image theory from these concepts, it is only necessary to find an expression for the image of a point object.

This conclusion may be obtained from the following argument. Consider an object consisting of two points located at x'_1 and x'_2 and let the intensity at these two points be given by $f(x'_1)$ and $f(x'_2)$, respectively. The object intensity $I_{obj}(x')$ may be expressed using delta functions as

$$I_{obj}(x') = f(x')\delta(x'-x'_1) + f(x')\delta(x'-x'_2). \tag{79}$$

Here δ is the Dirac delta function discussed in Section 3. The intensity image of one point may be denoted by $S(x)$, i.e.,

$$\delta(x') \rightarrow S(x), \tag{80}$$

where S(x) is called the impulse response or **point spread function**. The images of the two points can be written down immediately using the properties of linearity and stationarity. Thus,

$$f(x')\delta(x'-x'_1) \rightarrow f(x_1)S(x-x_1), \tag{81}$$

and

$$f(x')\delta(x'-x'_2) \rightarrow f(x_2)S(x-x_2); \tag{82}$$

$$\begin{gathered} f(x')\delta(x'-x'_1) + f(x')\delta(x'-x'_2) \\ \rightarrow f(x_1)S(x-x_1) + f(x_2)S(x-x_2). \end{gathered} \tag{83}$$

Similarly, if the object is consisted of a large set of points, i.e.,

$$I_{obj}(x') = \sum_{n=1}^{N} f(x')\delta(x'-x'_n), \tag{84}$$

then the output or image will be given by a sum of impulse responses, i.e.,

$$\sum_{n=1}^{N} f(x')\delta(x'-x'_n) \rightarrow \sum_{n=1}^{N} f(x_n)S(x-x_n). \tag{85}$$

We may define the image intensity as $I_{im}(x)$ and rewrite the above expression as an equation:

$$I_{im}(x) = \sum_{n=1}^{N} f(x_n)S(x - x_n). \tag{86}$$

Up to this point, we have considered only objects consisting of discrete points. However, the arguments are readily generalized to continuously varying objects as follows: Let $f(x")$ describe the intensity variation across a continuously varying scene. We may write $f(x")$ as a "sum of delta functions" by using the "combing" property of the delta function (see Section 3). Thus,

$$I_{obj}(x') = f(x') = \int f(x'')\delta(x' - x'')dx'. \tag{87}$$

The only difference between this representation of the object and the preceding one is that $I_{obj}(x')$ is now given by a continuous sum, an integral. Still the properties of linearity and stationarity allow us to write the output as a continuous sum of impulse responses, i.e.,

$$\int f(x'')\delta(x' - x'')dx'' \rightarrow \int f(x')S(x - x')dx'. \tag{88}$$

Hence, denoting the image by $I_{im}(x)$, we write

$$I_{im}(x) = \int f(x')S(x - x')dx' \tag{89}$$

Equation (89) is, of course, exactly analogous to Equation (86) where the integral (a continuous summation) replaces the discrete summation. Equation (89) is, in fact, the starting point for the analysis of any linear stationary system.

It is clear from Eq. (89) that if we can obtain an expression for the image of a point object, i.e., the point spread function, we can determine the image I_{im}, by convolving the object distribution $f(x')$ with the point spread function[2] We may then use this answer to describe the image of an arbitrary object by using Eq. (89). To obtain an expression for the impulse response, we must determine the wave-optical solution for the image of a point object.

5.3 Image of a point object

Starting from an optical field $\Psi_{obj}(x')$ at the object plane located at a distance z from a lens with aperture $A(\xi)$ (see Fig. 16), we intend to calculate the field distribution $\Psi_{im}(x)$ in the image plane at a distance z' on the other side of this lens.

Here we limit our consideration to a **one**-dimensional situation but this analysis is easily extended to the two-dimensional case. Let r be the distance

[2]Through-out this text, the terms "impulse response," "point spread function," and "point diffraction pattern" are used interchangeably.

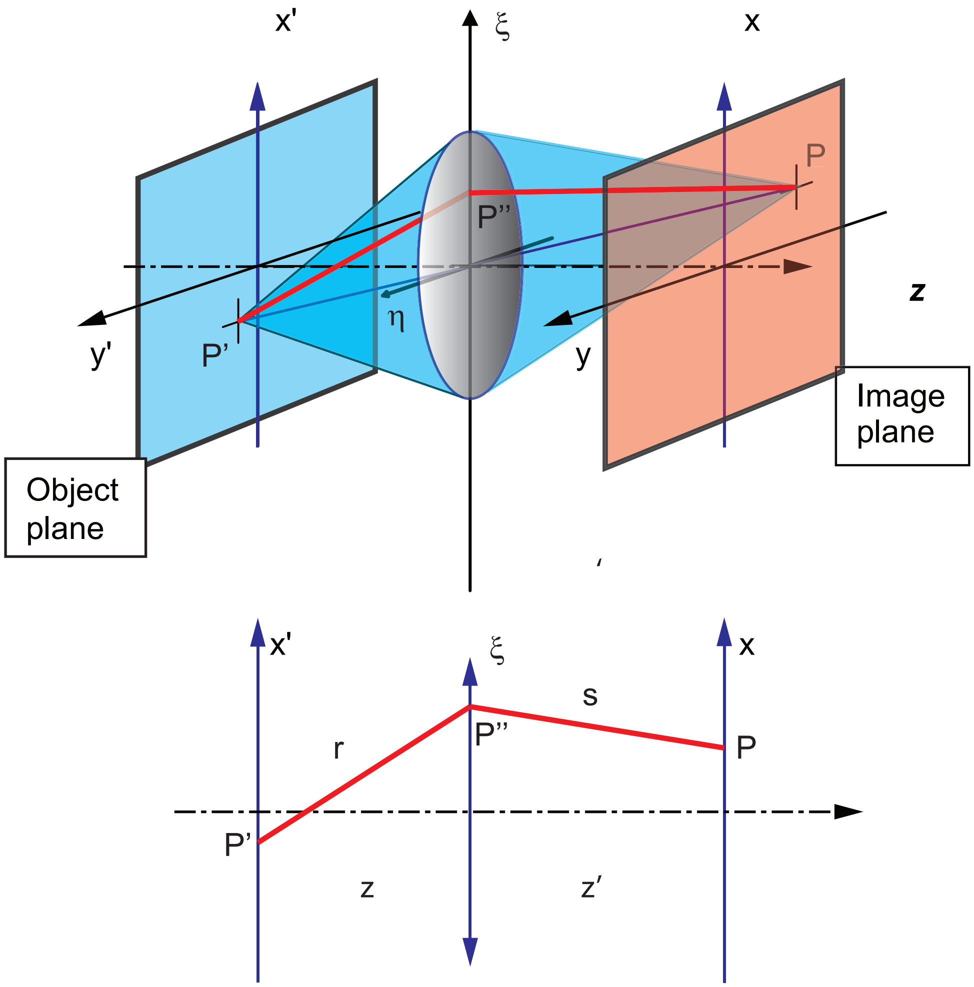

Figure 16: point object and point image across a single lens

$\overline{P'P''}$ from the object plane to the lens plane or aperture plane and s the distance $\overline{P''P}$ from the aperture plane to the image plane. Furthermore, in Section 1, the effect of a lens was shown to be a phase factor $\exp(-ik\xi^2/2f)$. We can describe this situation by the following integral representation

$$\Psi_{im}(x) = K_1 \int (\int \Psi_{obj}(x') \frac{e^{ikr}}{r} A(\xi) dx') exp\left(\frac{-ik\xi^2}{2f}\right) \frac{e^{iks}}{s} d\xi; \tag{90}$$

where the first integral containing e^{ikr}/r represents a spherical wave propagating from an object point to the lens plane, and the second integral containing e^{iks}/s is the spherical wave from each point on the wavefront immediately after the lens. As already mentioned the phase factor $exp\left(\frac{-ik\xi^2}{2f}\right)$ represents the phase change caused by the lens.

For the distance r we obtain

$$r^2 = |\mathbf{x}'|^2 - 2\mathbf{x}' \cdot \xi + |\xi|^2 = R'^2 - 2\mathbf{x}' \cdot \xi + |\xi|^2, \tag{91}$$

where the coordiantes of the object point are characterized by

$$R'^2 = |x'|^2. \tag{92}$$

Similarly,

$$|s|^2 = |\mathbf{x}|^2 - 2\mathbf{x} \cdot \xi + |\xi|^2 = R^2 - 2\mathbf{x} \cdot \xi + |\xi|^2 \tag{93}$$

where the coordiantes of the image point are characterized by

$$R^2 = |x|^2 \tag{94}$$

As the distances r and s are much bigger than R or R' we can apply a binominal expansion to Eqs. (92) and neglect terms in second order. We obtain the following approximation for the distances r and s

$$\begin{cases} |r| \simeq R' - \frac{x'\xi}{R'} + \frac{\xi^2}{2R'}, \\ |s| \simeq R - \frac{x\xi}{R} + \frac{\xi^2}{2R}. \end{cases} \quad \cdots \tag{95}$$

Substituting these expressions into Eq. 90 we obtain

$$\begin{aligned} \Psi_{im}(x) = K_1 \frac{e^{ik(R+R')}}{RR'} &\iint \Psi_{obj}(x') A(\xi) \\ &\times exp\left[(\frac{-ik\xi^2}{2})\left(\frac{1}{R} + \frac{1}{R'} - \frac{1}{f}\right)\right] exp\left[ik\xi\left(\frac{x'}{R'} + \frac{x}{R}\right)\right] dx'd\xi. \end{aligned} \tag{96}$$

If $x' \ll f$ and $x' \ll z$ with similar constraints on x, then we have, essentially, paraxial optics. Combining the constant factors outside the integral,

$$\begin{aligned} \Psi_{im}(x) = K &\iint \Psi_{obj}(x') A(\xi) \\ &\times exp\left[(\frac{-ik\xi^2}{2})\left(\frac{1}{R} + \frac{1}{R'} - \frac{1}{f}\right)\right] exp\left[ik\xi\left(\frac{x'}{R'} + \frac{x}{R}\right)\right] dx'd\xi. \end{aligned} \tag{97}$$

where the prefactor K is

$$K = K_1 \frac{e^{ik(R+R')}}{RR'} \tag{98}$$

and the distances r and s have been approximated by the respective axial distances. The first exponential term goes to unity if

$$\frac{1}{z} + \frac{1}{z'} = \frac{1}{f}. \tag{99}$$

However, this is the lens equation already known from ABCD analysis and geometrical optics. Using this relation, we obtain

$$\Psi_{im}(x) = K \iint \Psi_{obj}(x') A(\xi) \exp\left[-ik\xi\left(\frac{x'}{z'} + \frac{x}{z}\right)\right] dx' d\xi. \tag{100}$$

Expanding the exponential term, we find

$$\Psi_{im}(x) = K \iint \Psi_{obj}(x') \exp\left(\frac{-ik\xi x'}{z'}\right) A(\xi) \exp\left(\frac{-ik\xi x}{z}\right) dx' d\xi. \tag{101}$$

As can be seen Eq. (101) contains two Fourier transforms. We calculate first the x' integration and obtain

$$\Psi_{im}(x) = K \int \tilde{\Psi}_{obj}\left(\frac{\xi}{z\lambda}\right) A(\xi) \exp\left(\frac{-ik\xi x}{z}\right) d\xi. \tag{102}$$

Here $\tilde{\Psi}_{obj}(\xi/z\lambda)$ is the Fourier transform of $\Psi_{obj}(x')$.

Since we are interested in the impulse response, we consider the object distribution to be a delta function, i.e.,

$$\Psi_{obj}(x') = \delta(x'). \tag{103}$$

Thus,

$$\tilde{\Psi}_{obj}\left(\frac{\xi}{z\lambda}\right) = \int \delta(x') \exp\left(\frac{-ik\xi x'}{z'}\right) dx' = 1. \tag{104}$$

Using this relation, Eq. (102) reduces to

$$\Psi_{im}(x) = K \int A(\xi) \exp\left(\frac{-ik\xi x}{z}\right) d\xi. \tag{105}$$

Equation 105 states that the amplitude distribution in the image plane of a point object is given by the Fourier transform of the aperture distribution function, i.e., the function describing the amplitude and phase variation introduced by passage through the lens. The constant K in Eq. (105) contains a spherical phase factor [see Eqs. (98), (92), and (**??**)], while this factor may be ignored at this point, it plays a crucial role in coherent imaging, holography and interferometry (see advanced chapters).

However, for incoherent imaging, we require an expression for the intensity impulse response $S(x)$. This is, of course, readily obtained from Eq. (105) by simply forming the squared modulus. Thus,

$$S(x) = \left| K \int A(\xi) \exp\left(\frac{-ik\xi x}{z}\right) d\xi \right|^2 . \tag{106}$$

Equations (105) and (106) are important results for imaging and will bbe used extensively in subsequent chapters. As a conclusion, we have shown that the impulse response can be calculated as a function of the aperture function $A(\xi)$.

5.4 Conclusions

In summary the main results are:

- An optical imaging system employing incoherent light may be considered to be a linear and stationary system in intensity.
- The detailed distribution of light in the image of an extended object can be calculated based on the point spread function i.e. the intensity distribution in the image plane of a point object.
- The distribution in the image of a point may be determined directly from the aperture function.

Eqs. (106) and (89) are sufficient to describe the image forming properties of an optical system. An alternative formulation and derivation which better lends itself to intuitive interpretation will be given later. In the following, we will use the convolution operation of Eqs. (89) and (106) to compute images for some simple examples in order to provide some insight into the implication of these results in the area of image quality prediction and evaluation.

5.5 Appendix: the relationship to geometrical optics

Equations (99) and (101) may be used to illustrate the relationship between wave optics and geometrical optics. First. it should be noted that the wave optics constraint to eliminate the quadratic phase error, e.g., Eq. (99), is precisely a consequence of the ABCD imaging condition. Later in the study of aberrations, the quadratic term in the Seidel expansion is referred to as the *focusing error*.

Furthermore, interesting relationships between the two different ways of modeling optical phenomena may be obtained by imposing the basic tenet of geometrical optics on Eq. (101). In geometrical optics, diffraction effects are completly ignored. We can accomplish that in this case by assuming that the diffraction pattern has no width, i.e., it is a Dirac delta function. This condition arises by assuming uniform illumination across the aperture in Eq. (100) and further assuming it to be infinitely wide. Under these conditions, Eq. (100) may be evaluated to yield

$$\psi_{im}(x) = K \int \psi_{obj}(x')\delta\left(\frac{x'}{\lambda z'} + \frac{x}{\lambda z}\right) dx'. \tag{107}$$

The integral in Eq. (107) may now be evaluated to yield

$$\psi_{im}(x) = K\psi_{obj}\left(-\frac{z}{z'}x\right). \tag{108}$$

Note that in this case the image is the same function as the object. That is, there has been no degradation due to diffraction. However, the negative argument indicates that the image is inverted. Also, the argument of the image function is scaled by the magnification, $m = -z/z'$; that is, the image is magnified by the ratio of the object distance to the image distance, all well known results from geometrical optics.

So far we have not discussed coherence. As a preliminary results without proof we would like to indicate the possiblity to extend this analysis for incoherently illuminated objects. If $I_{im}(x)$ denotes the intensity distribution in the image of such an object and $I_{obj}(x')$ the intensity distribution of the object, then the two are related by

$$I_{im}(x) = K' \int I_{obj}(x')S\left(x' + \frac{z}{z'}x\right) dx', \tag{109}$$

where $S(x)$ is the intensity impulse response as calculated from Eq. (106) and K' is a constant.